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Inverse Problems from Economics and Game Theory

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1 Introduction

We discuss open problems concerning inverses of theorems appearing in economics and game theory. We often finds the following Berge maximum theorem under convexity as a mathematical tools for optimal control problems in economics and game theory:

Theorem 1 [Berge] Let X be a subset of l-dimensional Euclidean space R^l and let Y be a subset of m-dimensional Euclidean space R^m. Let $u : X \times Y \to R$ be continuous and quasi-concave in its second variable, let $S : X \twoheadrightarrow Y$ be continuous and nonempty compact and convex-valued. Then, the correspondence $K : X \twoheadrightarrow Y$ defined by

$$K(x) = \{y \in S(x) : u(x,y) = \max_{z \in S(x)} u(x,z)\}, \quad x \in X \tag{1}$$

is upper semicontinuous and compact and convex-valued.

It is known that inverses of Theorem 1 hold (cf. [3], [5]) and we shall treat a related open inverse problem in Section 2.

Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space, $u : \Omega \times R^l_+ \to R_+$ a function with appropriate properties and $e \in L_1(\Omega, R^l_+)$. Then, for each $S \in \mathscr{F}$, define

$$v(S) \equiv \sup \left\{ \int_S u(\omega, x(\omega))\, d\mu(w) : x \in L_1(S, R^l_+),\ \int_S x\, d\mu = \int_S e\, d\mu \right\}. \tag{2}$$

The map v on $\mathscr{F}$ is called a *market game.* It is known that a market game is totally balanced and inner continuous at any $S \in \mathscr{F}$. (cf. [4]) We shall treat an open inverse problems concerning market games in Section 3.

2 Berge maximum theorem

In [3], the following inverse problem of Theorem 1 is considered:

> Let X be a subset of R^l and let Y be a convex subset R^m. Let $K : X \twoheadrightarrow Y$ be a nonempty compact convex-valued upper semi-continuous correspondence and let $S : X \twoheadrightarrow Y$ be a compact convex-valued continuous correspondence such that $K(x) \subset S(x)$ for $x \in X$. Then does there exist a continuous function $u : X \times Y \to R$ such that
>
> (i) $K(x) = \{y \in S(x) : u(x,y) = \max_{z \in S(x)} u(x,z)\}$ for $x \in X$;
>
> (ii) $u(x,y)$ is quasi-concave in y for $x \in X$?

and is obtained the following result:

Theorem 2 Let X be a subset of R^l. Let $K : X \twoheadrightarrow R^m$ be a nonempty compact convex-valued upper semicontinuous correspondence. Then there exists a continuous function $v : X \times R^m \to [0,1]$ such that

(i) $K(x) = \{y \in R^m : v(x,y) = \max_{z \in R^m} v(x,z)\}$ for any $x \in X$;

(ii) $v(x,y)$ is quasi-concave in y for any $x \in X$.

It is tried to generalize Theorem 2 to infinite dimensional case and a result is obtained in [5].

For a topological space X and a subset Y of a topological vector space, a correspondence $K : X \twoheadrightarrow Y$ is said to be *σ-selectionable* if there exists a sequence $\{K_n\}$ of continuous correspondences $K_n : X \twoheadrightarrow Y$ with compact convex values such that

(a) $K_{n+1}(x) \subset K_n(x)$ for any $x \in X$ and any $n \in \mathcal{N}$; and

(b) $K(x) = \bigcap_n K_n(x)$ for any $x \in X$.

It is known that an upper semicontinuous correspondence $K : X \twoheadrightarrow R^m$, $X \subset R^l$, with compact convex values is σ-selectionable and hence the following theorem obtained in [5] is a generalization of Theorem 2.

Theorem 3 Let X be a topological space, and Y a metric t.v.s. whose balls are convex, and $K : X \twoheadrightarrow Y$ a σ-selectionable map. Then there exists a continuous function $u : X \times Y \to [0,1]$ such that

(i) $K(x) = \{y \in Y : u(x,y) = \max_{z \in Y} f(x,z)\}$ for any $x \in X$; and

(ii) $u(x,y)$ is quasi-concave in y for any $x \in X$.

It is not known that the assumption of σ-selectionability of the correspondence K can be removed or not even in the case that X and Y are subsets of Banach spaces. Thus we have a conjecture:

Conjecture 1 Let X be a subset of a Banach space, Y a Banach space and $K : X \twoheadrightarrow Y$ a compact convex-valued upper semicontinuous correspondence. Then there exists a continuous function $u : X \times Y \to [0,1]$ such that

(i) $K(x) = \{y \in Y : u(x,y) = \max_{z \in Y} f(x,z)\}$ for any $x \in X$; and

(ii) $u(x,y)$ is quasi-concave in y for any $x \in X$.

3 Market games

Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space. A *game* v is a nonnegative real valued function, defined on the σ-field $\mathscr{F}$, which maps the empty set to zero. An *outcome* of a game v is a finitely additive real valued function α on $\mathscr{F}$ scuh that $\alpha(\Omega) = v(\Omega)$. For an outcome α of v, an integrable function f satisfying $\int_S f \, d\mu = \alpha(S)$ for all $S \in \mathscr{F}$ is said to be an *outcome density* of α with respect to μ. An outcome indicates outcomes to each coalitions while an outcome density designates outcomes to every players. The *core* of v is the set of outcomes α satisfying $\alpha(S) \geq v(S)$ for all $S \in \mathscr{F}$.

To every game v we associate an extended real number $|v|$ defined by

$$|v| = \sup \left\{ \sum_{i=1}^{n} \lambda_i v(S_i) : \sum_{i=1}^{n} \lambda_i \chi_{S_i} \leq \chi_\Omega \right\}, \tag{3}$$

where $n = 1, 2, \ldots$, $S_i \in \mathscr{F}$, λ_i is a real number. The notation χ_A denotes the characteristic function of a subset A of Ω. For a game v with $|v| < \infty$, we define two games $\overline{v}$ and $\hat{v}$ by

$$\overline{v}(S) = \sup\left\{\sum_{i=1}^{n} \lambda_i v(S_i) : \sum_{i=1}^{n} \lambda_i \chi_{S_i} \leq \chi_S\right\}, \quad S \in \mathscr{F}, \tag{4}$$

$$\hat{v}(S) = \min\left\{\alpha(S) : \alpha \text{ is additive}, \ \alpha \geq v, \ \alpha(\Omega) = |v|\right\}, \quad S \in \mathscr{F}, \tag{5}$$

following [6]. A game v is said to be *balanced* if $v(\Omega) = |v|$, *totally balanced* if $v = \overline{v}$ and *exact* if $v = \hat{v}$, respectively. It is proved in [6] that the core of a game is nonempty if and only if it is balanced, every exact game is totally balanced, and every totally balanced game is balanced.

A game v is said to be *monotone* if $S \subset T$ implies $v(S) \leq v(T)$ for any S and T in $\mathscr{F}$. A game v is said to be *inner continuous* at S in $\mathscr{F}$ if it follows that $\lim_{n\to\infty} v(S_n) = v(S)$ for any nondecreasing sequence $\{S_n\}$ of measurable sets such that $\bigcup_{n=1}^{\infty} S_n = S$. Similarly, a game v is said to be *outer continuous* at S in $\mathscr{F}$ if it follows that $\lim_{n\to\infty} v(S_n) = v(S)$ for any nonincreasing sequence $\{S_n\}$ of measurable sets such that $\bigcap_{n=1}^{\infty} S_n = S$. A game v is *continuous* at S in $\mathscr{F}$ if it is both inner and outer continuous at S.

We denote utilities of players by a Carathéodory type function u defined on $\Omega \times R^l_+$ to R_+, where R^l_+ denotes the nonnegative orthant of the l-dimensional Euclidean space R^l, and R_+ is the set of nonnegative real numbers. The nonnegative number $u(\omega, x)$ designates the density of the utility of a player ω getting goods x. We always use the ordinary coordinatewise order when having concern with an order in R^l_+. We suppose that the function $u : \Omega \times R^l_+ \to R_+$ satisfies the conditions:

1. The function $\omega \mapsto u(\omega, x)$ is measurable for all $x \in R^l_+$;
2. The function $x \mapsto u(\omega, x)$ is continuous, concave, nondecreasing, and $u(\omega, 0) = 0$, for almost all ω in Ω;
3. $\sigma \equiv \sup\{u(\omega, x) : (\omega, x) \in \Omega \times B_+\} < \infty$, where $B_+ = \{x \in R^l_+ : \|x\| \leq 1\}$, and $\|x\|$ denotes the Euclidean norm of $x \in R^l_+$.

For any set S in $\mathscr{F}$, the set of integrable functions on S to R^l_+ is denoted by $L_1(S, R^l_+)$. We take an element e of $L_1(\Omega, R^l_+)$ as the density of initial endowments for the players. For any S in $\mathscr{F}$, define

$$v(S) \equiv \sup\left\{\int_S u(\omega, x(\omega))\, d\mu(w) : x \in L_1(S, R^l_+), \ \int_S x\, d\mu = \int_S e\, d\mu\right\}. \tag{6}$$

The set function v defined above is called a *market game* derived from the market $(\Omega, \mathscr{F}, \mu, u, e)$.

The following theorem is proved in [4]:

Theorem 4 The market game defined above is totally balanced and inner continuous at every $S \in \mathscr{F}$.

Every exact game which is continuous at Ω, equivalently inner continuous at Ω, is continuous at every S in $\mathscr{F}$ according to [6], but it is known that a market game is not necessarily continuous at every $S \in \mathscr{F}$. Thus we are interested in the following conjecture as an inverse problem of Theorem 4 to understand the difference between totally balanced games and exact games.

Conjecture 2 A totally balanced game that is inner continuous at any S in $\mathscr{F}$ is a market game, that is, a game derived from a market.

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